Practical considerations in the control of chaos

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(Received 21 January 1994)

Unstable periodic orbits in certain chaotic systems may be stabilized via small perturbations of a control parameter. Stabilization using linear feedback has been achieved in both simulations and physical experiments. Not all chaotic systems can be controlled easily or well, and the effectiveness of proposed control algorithms depends strongly on mathematical properties of the chaotic behavior. Practical considerations are discussed that affect the robustness of local linear control strategies, with emphasis on the range of feedback gains which can stabilize the linearized map.

PACS number(s): 05.45.+b, 06.70.Td

Many nonlinear dynamical systems exhibit chaotic oscillations. Recently, a general approach to controlling chaos in physical systems has been proposed, based on the existence of unstable periodic orbits within the chaotic attractor. The original technique, due to Ott, Grebogi, Yorke [1], relies on the local linearization of a surface of section map near an unstable fixed point. The Ott-Grebogi-Yorke (OGY) method consists of applying linear feedback control in the vicinity of such a fixed point.

In principle, the OGY method is applicable to any physical system, even without an accurate mathematical model, as long as iterates on a surface of section can be obtained. In several experiments, OGY control has been successful in stabilizing periodic motion. Ditto, Rauseo, and Spano [2] stabilized period-1 and period-2 oscillations of a magneto-elastic ribbon. Hunt [3] used a simplified analog version (simple proportional feedback) to control high-period orbits in an electronic circuit: the diode resonator. Experimental success was also reported for control of chaos in cardiac tissue [4], in an electrochemical cell [5], in the Belousov-Zhabotinsky reaction [6], and in diode and NMR lasers [7,8]. The concept has been extended, with results from simulations, to control transient chaos [9], to stabilize chaotic scattering [10], and to synchronize identical but distinct chaotic systems [11].

The success of these experiments and simulations notwithstanding, local linear control is not practical for every physical chaotic system. A number of important properties, which can dictate whether OGY control or similar methods will work well, are discussed below. These properties include the degree of instability of unstable orbits, the strength of governing nonlinearities, the dimension of the chaotic attractor, the sensitivity of the unstable orbits to changes in accessible parameters, and the sensitivity of the control scheme to the precise value of the feedback gain.

OGY control. The OGY approach is explained in their original paper [1] and also in Refs. [12] and [13]. A short description is given here for convenience. Chaotic trajectories approach arbitrarily close to many unstable saddle-type periodic orbits [14], corresponding to unstable fixed points of the associated surface of section, or Poincaré, map: $\mathbf{x}_{n+1} = \mathbf{P}(\mathbf{x}_n)$. Three-dimensional continuous systems with two-dimensional (2D) Poincaré maps will be considered here, though the stabilization algorithms may, in principle, be generalized to N dimensions. In a 2D map, unstable saddle points are associated with one unstable eigenvalue ($|\lambda_u| > 1$) and its corresponding eigenvector $\mathbf{e}_u$, as well as one stable eigenvalue ($|\lambda_s| < 1$) and its eigenvector $\mathbf{e}_s$. In Fig. 1 a caricature is shown of an unstable fixed point in a 2D nonlinear map, along with its unstable and stable manifolds and their tangent approximations. The local linear approximation is represented completely by these tangent eigenvectors and the corresponding eigenvalues of the Jacobian matrix A. The idea behind OGY control is to nudge iterates of the map onto the stable eigenvector.

If the Poincaré map $\mathbf{P}$ is made explicitly dependent on a parameter p, the map can be written as

$$\mathbf{x}_{n+1} = \mathbf{P}(\mathbf{x}_n, p). \tag{1}$$

Linearizing about the fixed point $\mathbf{x}_f$, letting $\boldsymbol{\xi} = \mathbf{x} - \mathbf{x}_f$, and also letting the nominal value of p be $p = 0$, we obtain the local approximation

$$\boldsymbol{\xi}_{n+1} \approx A\,\boldsymbol{\xi}_n + \mathbf{h}\,\delta p, \tag{2}$$

where A is the Jacobian matrix, $A_{ij} = \partial \mathbf{P}_i/\partial \xi_j$, and $\mathbf{h}$ is the correction resulting from the small perturbation δp: $\mathbf{h} = \partial \mathbf{P}/\partial p$. Equation (2) is valid in some neighborhood of $\boldsymbol{\xi} = \mathbf{0}$; the size of the neighborhood depends on the strength of the nonlinearity in $\mathbf{P}$.

The unstable contravariant eigenvector $\mathbf{f}_u$ is defined by the conditions $\mathbf{e}_u \cdot \mathbf{f}_u = 1$ and $\mathbf{e}_s \cdot \mathbf{f}_u = 0$. The component of the linearized map along $\mathbf{f}_u$ (the "unstable component") obeys the relation

$$\xi^u_{n+1} = \lambda_u\, \xi^u_n + h_u\, \delta p, \tag{3}$$

where λ_u is the unstable eigenvalue of A. The stable component can be similarly isolated.

In the OGY approach, if the nth iterate of the map is sufficiently near the fixed point, the control parameter is perturbed in proportion to the unstable component: $\delta p_n = \alpha \xi^u_n$. This leads to new local dynamics

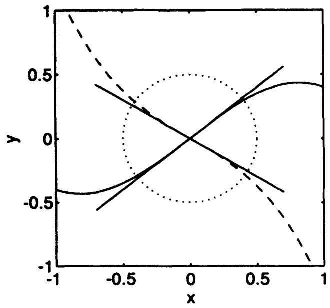

FIG. 1. Schematic diagram of a fixed point (0,0) in a nonlinear map. The stable and unstable manifolds of the fixed point are shown, along with the eigenvectors of the linearized system. The dotted circle indicates the region where the linear approximation may be useful.

$$\begin{bmatrix} \xi^u_{n+1} \\ \xi^s_{n+1} \end{bmatrix} = \begin{bmatrix} (\lambda_u + \alpha h_u) & 0 \\ \alpha h_s & \lambda_s \end{bmatrix} \begin{bmatrix} \xi^u_n \\ \xi^s_n \end{bmatrix} \tag{4}$$

whose new eigenvalues are $s_1 = \lambda_u + \alpha h_u$ and $s_2 = \lambda_s$.

Noting that the stable eigenvalue is not affected by the control, the unstable component is examined. The value of s_1 is determined by the choice of α. OGY suggest that α be chosen so that the unstable component is eliminated in one iteration, i.e., by setting s_1 equal to zero. This is accomplished by choosing $\alpha = -\lambda_u/h_u$. The nth parameter perturbation δp_n is then

$$\delta p_n = (-\lambda_u/\mathbf{f}_u \cdot \mathbf{h})[\mathbf{f}_u \cdot (\mathbf{x}_n - \mathbf{x}_f)], \tag{5}$$

where the vector dot products are shown explicitly.

In fact stabilization is attained for many possible values of α. As pointed out by Romeiras *et al.* [12], the choice of α is an example of "pole placement," a classical technique in control systems theory. The necessary condition for stabilization is that $|s_1| < 1$. This condition sets bounds on the gain α as a function of the unstable eigenvalue λ_u:

$$(-\lambda_u - 1)/h_u < \alpha < (-\lambda_u + 1)/h_u \ . \tag{6}$$

These stability boundaries are plotted in Fig. 2. The width of the stability boundary is a good measure of the robustness of the control to errors in the estimates of eigenvalues, eigenvectors, and sensitivity to control.

Consecutive difference control. The OGY method relies on accurate knowledge of the location of the map's unstable fixed point. If the system changes, the fixed point will change and the control strategy will become invalid. The OGY method also relies on the chaotic motion itself to bring the trajectory close to the unstable periodic orbit. Bielawski, Derozier, and Glorieux [7] proposed a closely related control scheme that does not rely on knowledge of the location of the fixed point and hence can be used if the system evolves. By using the differences between consecutive iterates of the map, instead of the differences between each iterate and the fixed point, their approach can locate and stabilize unstable periodic orbits both within and outside chaotic regimes.

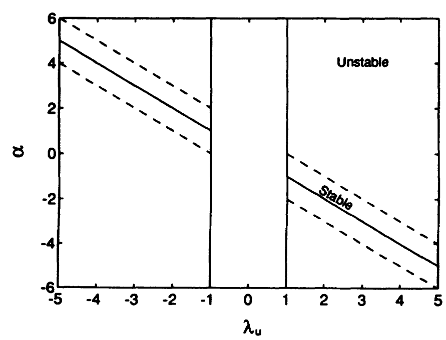

FIG. 2. Optimal gain value α for OGY control (solid line) and the maximum and minimum gains (dashed line) for stabilization, plotted versus the unstable eigenvalue λ_u ($|\lambda_u| > 1$).

Setting the control perturbation proportional to the difference between the unstable components of consecutive points on the Poincaré section, $\delta p_n = \alpha(x^u_n - x^u_{n-1})$, a modified dynamical system is obtained. Writing $y_n = x_{n-1}$, the unstable dynamics in the linear neighborhood of the fixed point become

$$\begin{bmatrix} x^u_{n+1} \\ y^u_{n+1} \end{bmatrix} = \begin{bmatrix} (\lambda_u + \alpha h_u) & -\alpha h_u \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x^u_n \\ y^u_n \end{bmatrix} + \mathbf{c}. \tag{7}$$

The eigenvalues of the controlled system are $s_1, s_2 = (\lambda_u + \alpha h_u)/2 \pm \sqrt{(\lambda_u + \alpha h_u)^2 - 4\alpha h_u}/2$. Investigation of these eigenvalues reveals that this algorithm, identified here as consecutive difference (CD) control, cannot stabilize fixed points with $\lambda_u > 1$ or $\lambda_u < -3$. Bielawski, Derozier, and Glorieux recognized this shortcoming and proposed a further modification.

Alternating CD control. If the control is not applied every cycle, but every second cycle, the parameter perturbation can be written

$$\delta p_{2n} = \alpha(x^u_{2n} - x^u_{2n-1}), \quad \delta p_{2n+1} = 0. \tag{8}$$

Again setting $y_n = x_{n-1}$, the modified dynamics become

$$\begin{bmatrix} x^u_{2n+2} \\ y^u_{2n+2} \end{bmatrix} = \begin{bmatrix} \lambda_u(\lambda_u + \alpha h_u) & -\lambda_u \alpha h_u \\ (\lambda_u + \alpha h_u) & -\alpha h_u \end{bmatrix} \begin{bmatrix} x^u_{2n} \\ y^u_{2n} \end{bmatrix} + \mathbf{c}. \tag{9}$$

Now the eigenvalues of the controlled system are $s_1 = 0$ and $s_2 = \lambda_u^2 + (1 - \lambda_u)\alpha h_u$. Paradoxically, the application of control only every other cycle allows (in theory) stabilization for any value of λ_u. The stability bounds on the feedback gain α are

$$(\lambda_u^2 - 1)/[(\lambda_u - 1) h_u] < \alpha < (\lambda_u^2 + 1)/[(\lambda_u - 1) h_u]. \tag{10}$$

These stability boundaries are shown in Fig. 3. Clearly, while it is always possible to find an α which will stabilize the system, as $|\lambda_u|$ becomes large the stability of the system becomes very sensitive to the choice of gain.

Recursive CD control. Another method of expanding the theoretical range of consecutive difference control is to make each control perturbation explicitly dependent on the previous perturbation. Bielawski, Derozier, and Glorieux mention this possibility, though they do not implement it; it was proposed in a different context by Dressler and Nitsche [13].

Letting $\delta p_n = \alpha(x^u_n - x^u_{n-1}) + \beta\,\delta p_{n-1}$, the modified dynamics become

$$\begin{bmatrix} x^u_{n+1} \\ \delta p_{n+1} \end{bmatrix} = \begin{bmatrix} \lambda_u & h_u \\ \alpha(\lambda_u - 1) & \alpha h_u + \beta \end{bmatrix} \begin{bmatrix} x^u_n \\ \delta p_n \end{bmatrix} + \mathbf{c} \tag{11}$$

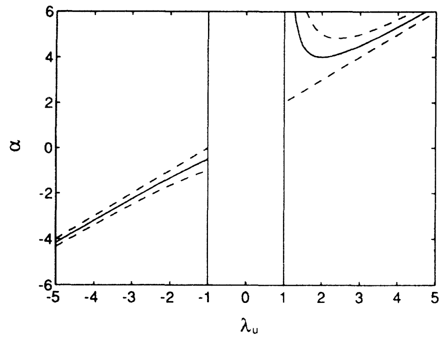

FIG. 3. Optimum values (solid lines) and stability boundaries (dashed lines) for the control gain α as a function of λ_u in the alternating version of consecutive difference control.

with the new eigenvalues $s_1, s_2 = (\lambda_u + \alpha h_u + \beta)/2 \pm \sqrt{(\lambda_u + \alpha h_u + \beta)^2 - 4\alpha h_u - 4\beta\lambda_u}/2$.
The stability conditions are

$$[(1+\beta)(1+\lambda_u)]/-2h_u < \alpha < (1-\beta\,\lambda_u)/h_u \quad (12)$$

and

$$(3+\lambda_u)/(\lambda_u - 1) < \beta < 1, \quad \lambda_u < 1$$
$$(3+\lambda_u)/(\lambda_u - 1) > \beta > 1, \quad \lambda_u \geq 1. \quad (13)$$

The stability boundaries for α and β as a function of λ_u are shown in Fig. 4.

It might be inferred from the range of physical systems which have been stabilized that the methods described above will work quite well for any chaotic system. On the other hand, the successful experimental examples of control of chaos have several important characteristics in common. First, all are truly examples of *low-dimensional* chaos. That is not to say that the physical systems are inherently low-order systems (several are infinite-degree-of-freedom systems), but that their chaotic dynamics lie in a two-to-three-dimensional subspace. In fact, each of the well-known controlled examples are strongly dissipative, so that their Poincaré sections are "quasi-unidimensional" [2,3,5,7]. This one dimensionality allows control to be based on measurements of only one state variable and allows easy identification of fixed points and unstable eigenvalues [15]. Second, the fixed points to be stabilized were only weakly unstable ($|\lambda_u| < 2$), at least in the references which supplied the unstable eigenvalues

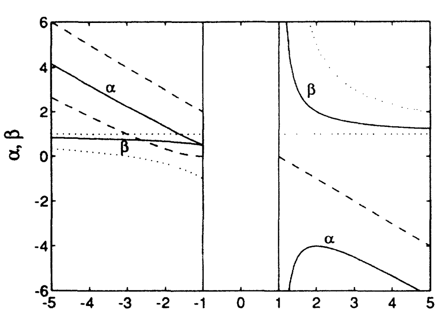

FIG. 4. Optimum values (solid lines) and stability boundaries of the feedback gain α (dashed boundaries) and the recursive gain β (dotted boundaries) in the recursive version of consecutive difference control.

[2,5,7]. Finally, the maps appeared only weakly nonlinear near the chosen unstable fixed points [2,5], enabling the approximation of local linearity to hold quite well in a relatively large region.

Several properties of the basic dynamics, as well as the experimental implementation, of nonlinear systems are discussed below. These characteristics affect the ability of unstable periodic orbits to be stabilized by local linear methods.

Strongly nonlinear dynamics. The stronger the nonlinearity of the dynamics near the fixed point, the smaller the region in which the local linearization holds true (see Fig. 1). The radius of the "linear region" in state space is proportional to $1/\sqrt{c_2}$, where c_2 is the coefficient of the largest second-order nonlinear term in the Taylor series expansion of the map. When c_2 is large, the region where the control law is valid may be very small. Small linear regimes will also increase the difficulty of estimating a *correct* local linear approximation. The error in a linear fit of nonlinear data increases with the strength of the nonlinearity and with the radius of the cluster of points used to obtain the linear approximation. The difficulty of estimating a good linear approximation and the inaccuracy inherent even in a perfect linearization both tend to increase the time to achieve control and to decrease the robustness of the control algorithm.

Strongly unstable dynamics. The size of the required control perturbation in all the methods discussed above is directly proportional to the magnitude of λ_u [see, for example, Eq. (5)]:

$$|\delta p_n| \propto |\lambda_u|. \quad (14)$$

In addition, the more unstable the dynamics, the more quickly the map departs from the linear neighborhood of the fixed point:

$$\tau \propto 1/\ln|\lambda_u|, \quad (15)$$

where τ denotes the number of iterations to leave the linear region. Large magnitudes of λ_u also magnify the effect of errors in the linear approximation (see below and in [16]).

Effectiveness of control parameter. The control parameter p must be accessible. It must be perturbed quickly in comparison to the system dynamics. Otherwise, a perturbation chosen for its effect on a given surface of section will act on a shifted section with quite different local dynamics. The parameter must also have a strong effect $\mathbf{h}$ on local dynamics, particularly along the unstable manifold. The size of the required perturbation is inversely proportional to the quantity $h_u = \mathbf{f}_u \cdot \mathbf{h}$ [see Eq. (5), for example]. If the unstable dynamics are relatively insensitive to the control parameter, large perturbations will be required to accomplish small local changes.

Estimation error. The quantities $\mathbf{x}_f$, $\mathbf{f}_u$, and $\mathbf{h}$ in Eq. (5) (for example) must all be estimated from experimental data in order to compute the feedback gain α. (An exception is in fast systems, where the gain may be varied in real time to find a value which "locks on" to an orbit, as, for example, in Ref. [3].) Uncertainties in estimation may lead to inaccurate and ineffective calculated gains, especially if the magnitude of the unstable eigenvalue is large.

Petrov, Peng, and Showalter showed the effect of errors on the stabilization of a 1D map [16]. Their discussion can be generalized to the N-dimensional algorithms, as long as only one eigenvalue is unstable. For example, in OGY control, an error $\boldsymbol{\epsilon}$ in the estimate of the fixed point ($\mathbf{x}'_f = \mathbf{x}_f - \boldsymbol{\epsilon}$) will lead to stabilization of the point

$$\mathbf{x}^* = \mathbf{x}_f + \lambda_u \left(\mathbf{f}_u \cdot \boldsymbol{\epsilon}\right) \mathbf{f}_u, \tag{16}$$

assuming other quantities are valid and the original linearization holds. Large $|\lambda_u|$ will place $\mathbf{x}^*_f$ outside the linear region unless the error $\boldsymbol{\epsilon}$ is small [16].

The permissible error in the control gain α is shown directly by the width of the stability regions shown in Figs. 2–4. Errors in estimating λ_u, $\mathbf{f}_u$, and $\mathbf{h}$ all contribute to error in α. In particular, errors in the magnitude and direction of both $\mathbf{f}_u$ and $\mathbf{h}$ lead to an error ϵ in h_u, which appears in the denominator of all the gain equations. Letting the incorrect value $h'_u = h_u(1+\epsilon)$ and using $1/(1+\epsilon) \approx 1 - \epsilon$, the resulting error in the gain α is

$$\epsilon_\alpha \approx F(\lambda_u)\, \epsilon, \tag{17}$$

where the magnitude of $F(\lambda_u)$ is an increasing function of $|\lambda_u|$ [see Eqs. (6), (10), and (12)].

Time to achieve control. The characteristic time scale of the original system is very important. A control scheme that takes thousands or millions of cycles to lock on is more practical in systems operating at 10 kHz than 1 Hz. The time to achieve control is otherwise controlled by the dimension of the chaotic attractor D and the size of the linear neighborhood surrounding the relevant fixed point [12]. Suppose that a particular local linear approximation holds true in a sphere of radius 1/10 relative to the size of a three-dimensional attractor. That neighborhood will probably be visited about 100 times less frequently than a linear neighborhood occupying 1/10 of a one-dimensional set.

Noise. The noise level in a system can also be thought of as a radius of uncertainty in state space (Fig. 1). If this radius is larger than the size of the region where the local linear approximation remains accurate, then the linear control schemes described above will not maintain periodic behavior. A noise-related error $\boldsymbol{\epsilon}$ in the measurement of the point $\boldsymbol{\xi}_n$ ($\boldsymbol{\xi}'_n = \boldsymbol{\xi}_n + \boldsymbol{\epsilon}$) leads to an error in the control applied during the next iteration. Substituting $\boldsymbol{\xi}'_n$ into Eq. (2), the expression

$$\boldsymbol{\xi}_{n+1} = A\,\boldsymbol{\xi}_n + \mathbf{h}\,\delta p + \alpha\,\mathbf{h}\left(\mathbf{f}_u \cdot \boldsymbol{\epsilon}\right) \tag{18}$$

is obtained, where the last term on the right-hand side is due to measurement error. Large gains (resulting from large $|\lambda_u|$), thus tend to amplify the inaccuracy of measurements. If $\boldsymbol{\xi}_{n+1}$ falls outside the linear region, control may be lost.

Robustness of control. If there is little margin for error in the choice of feedback gain for a given control scheme, then the control will not be very robust to estimation errors or deviation from linear behavior. The width of the stable regions in Figs. 2–4 reflects the robustness of the control for a given value of the unstable eigenvalue. In particular, the range of effective gains for the alternating CD control becomes very small as the magnitude of λ_u increases.

Time-delay coordinates. If only one state variable is accessible for measurement and local dynamics are not close to being one-dimensional, time-delay coordinates may be used to describe observed data [17]. Dressler and Nitsche [13] showed that using time-delay coordinates in a control scheme introduces a complication however. The Poincaré map in delay coordinates explicitly depends not only on the current iteration's parameter value p_n, but also on the parameter value during the previous cycle p_{n-1} [13]. The linear approximation of the map then takes the form $\boldsymbol{\xi}_{n+1} \approx A\,\boldsymbol{\xi}_n + \mathbf{h}\,\delta p_n + \mathbf{g}\,\delta p_{n-1}$, where $\mathbf{g}$ represents the correction due to a perturbation in the previous cycle $\mathbf{g} = \partial\mathbf{P}/\partial p_{n-1}$. The vector $\mathbf{g}$ is more difficult to estimate than $\mathbf{h}$ since the parameter p must be alternately perturbed and reset [13].

The control scheme devised by Ott, Grebogi, and Yorke, and modified by subsequent authors, is a remarkable application of nonlinear dynamical systems theory. In practice, current versions will work better on systems whose Poincaré sections are nearly one-dimensional, with weakly nonlinear and weakly unstable local dynamics. Necessary properties of the physical experiment include an accessible parameter that can be changed quickly and which affects local dynamics strongly. In applying control, the effects of using time-delay coordinates may be significant. Reducing experimental noise is crucial. A control strategy that has as large a margin of error as possible can be selected on the basis of linear stability criteria. The range of gains that can stabilize a given fixed point is a good measure of the robustness of the control algorithm.

This research was supported by the National Science Foundation Grant No. NSF-MSS-92-09886. E.H. Dowell, D. Gauthier, and J. Socolar provided useful discussions.

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